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A new exactly solvable case of an O(n)-model on a hexagonal lattice

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Abstract. A new solvable case of O(n)-model on a hexagonal lattice is presented. It is shown that the model has a phase transition at $n = 2$. This phase transition can be interpreted as phase transition between percolating and non-percolating phases.

1. Introduction

Lately, we have observed an upsurge of interest in the O(n)-model of statistical physics [1–4]. The fluctuating parameters of the model are n-dimensional vectors with fixed norm $s^2 = n$. With each vertex of a hexagonal lattice we associate such a parameter. The partition function is defined as follows:

$$Z_{N_0}^{O(n)}(T) = \int \prod_{i\in\mathscr{L}} \mathrm{d}s_i \prod_{\langle ij\rangle} \left(1 + \frac{1}{T} s_i s_j\right). \tag{1}$$

Here T is the temperature, $\langle ij\rangle$ are the neighbouring vertices and the periodic boundary conditions are provided.

The high-temperature expansion of this partition function is defined by the partition function of the loop model:

$$Z_{N_0}^{O(n)}(T) = \sum_C \left(\frac{1}{T}\right)^{L(C)} n^{N(C)} = T^{-N_0} Z_{\text{loop}}(T, n). \tag{2}$$

Here the sum runs over all coverings of the lattice $\mathscr{L}$ by closed non-intersecting and non-self-intersecting contours (we will call such coverings loop coverings); $L(C)$ is the total length of all the loops, $N(C)$ is the number of loops. The partition function (1) can be analytically continued to non-integer values of n.

Nienhuis [1] found that if

$$n = 2 - (2 - T^2)^2 \tag{3}$$

model (2) is equivalent to the Potts model [5] on a triangular lattice:

$$Z_{\text{Potts}} = \sum_{\{\sigma\}} \exp\left(J \sum_{\langle ij\rangle} \delta_{\sigma_i\sigma_j}\right) \tag{4}$$

with $e^J = 0$, $\sigma = 1, \ldots, n$.

As was shown by Baxter [5], model (2) on line (3) is solved by the Bethe ansatz method. In the appendix we demonstrate that this model is also equivalent to a known

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integrable lattice model [6, 7] on a square lattice connected with the affine Lie algebra $A_2^{(2)}$.

In the present work a new integrable case of model (2) is found. It is shown that this model at $T=0$ is equivalent to the known integrable model connected with the affine Lie algebra $A_2^{(1)}$. In this case, partition function (2) becomes

$$Z_{\text{loop}}(0, n)=\sum_{\hat{C}} n^{N(\hat{C})} \tag{5}$$

where $\hat{C}$ are the dense loop coverings of the lattice. In such coverings, each vertex of the lattice lies on some loop.

This model can be considered as a model of percolation (see, for example, [8]). Indeed, coverings giving a maximal contribution to a partition function (1) for large n contain an infinite number of loops (at $N_0=\infty$). These configurations correspond to the absence of percolations (for example along loops). As $n\to 0$, typical configurations contain a finite number of loops (at $N_0=\infty$). Of course, it is natural to interpret these configurations as percolations (for example along loops). One can suppose that there exists a phase transition in this model at some finite $n=n_c$ dividing the percolating phase from the non-percolating one. As we shall show, $n_c=2$.

One can give another interpretation of model (1) by comparing it with the loop soup model [4] describing the RVB state in high-temperature superconductivity. In this interpretation the absence of percolation corresponds to the superconducting phase.

In section 2 of the present work the partition function (1) is rewritten through the partition function of the already known exactly solvable [9, 10] model on a square lattice. The partition function of this model in the thermodynamic limit is calculated in section 3. Critical behaviour of the model is discussed in section 4.

2. Equivalence of the proposed model to the already known model on a square lattice

Let us consider the vertex model on a square lattice of size $N_0=N\times M$ with three states and with cyclic boundary conditions. The states in this model are placed at the edges of the lattice. With each vertex we associate a Boltzmann weight, which depends on the states of the nearest edge. For example, with the vertex on figure 1 we associate the weight W_{jl}^{ik}. Let us choose a matrix W_{jl}^{ik} in the following form:

$$\begin{aligned} W_{jl}^{ik} = {} & \sum_{p=1}^{3} \delta_{ip}\delta_{lp}\delta_{kp}\delta_{jl}(qz-q^{-1}z^{-1}) + \sum_{p\neq p'}^{3} \delta_{ip}\delta_{lp'}\delta_{kp'}\delta_{jp}(z-z^{-1}) \\ & +(q-q^{-1})\left(z\sum_{a>b}^{3}\delta_{ia}\delta_{la}\delta_{jb}\delta_{kb} - z^{-1}\sum_{a<b}^{3}\delta_{ia}\delta_{jb}\delta_{la}\delta_{kb}\right) \\ & i, j, k, l=1,2,3. \end{aligned}$$

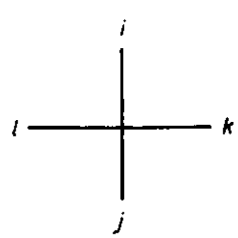

Figure 1. States on the lattice.

We shall consider (W^{ij}_{kl}) as the matrix acting in $\mathbb{C}^3\otimes\mathbb{C}^3$ (where $\mathbb{C}^3\otimes\mathbb{C}^3$ is the product of the space of the states on horizontal edges by the space of the states on vertical edges) and denote it as $R(z,q)(R(z,q)e_i\otimes e_j=\Sigma^3_{k,l=1}W^{kl}_{ij}e_k\otimes e_l)$. This matrix satisfies the Yang–Baxter equation [11]

$$R_{12}(z,q)R_{13}(zx,q)R_{23}(x,q)=R_{23}(x,q)R_{13}(zx,q)R_{12}(z,q).$$

Here, the subscripts indicate how $R(z,q)$ is embedded into $\mathrm{END}(\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3)$.

The transfer matrix of the model acts in $\mathscr{H}_N=(\mathbb{C}^3)^{\otimes N}$ and is of the form

$$t(z)=\mathrm{tr}_a(R_{a1}(z,q)\dots R_{aN}(z,q)). \tag{6}$$

The partition function of model (2) is equal to the trace of the Mth power of the transfer matrix:

$$\tilde{Z}_{N\times M}(z,q)=\mathrm{tr}_{\mathscr{H}_N}(t(z)^M). \tag{7}$$

Let us explain now the connection between this model and the model defined by (5). To this end we enumerate the diagonals of the lattice (figure 2). With each diagonal we associate the space of states $\mathbb{C}^3$.

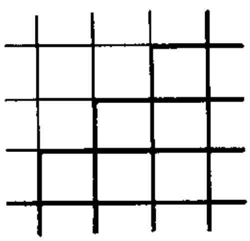

Figure 2. Diagonals in the lattice.

Associate the space $V_\alpha\simeq\mathbb{C}^3$ with the αth diagonal of the lattice as is shown in figure 2. The matrix of Boltzmann weights can be considered as acting in $V_\alpha\otimes V_{\alpha+1}\simeq\mathbb{C}^3\otimes\mathbb{C}^3$ in the following way:

$$\hat{W}e_i\otimes e_j=PR(z,q)e_i\otimes e_j$$

where P is the permutation matrix in $\mathbb{C}^3\otimes\mathbb{C}^3(P(e_i\otimes e_j)=e_j\otimes e_i)$.

The vectors

$$\begin{aligned}\bar{e}_i&=\frac{1}{\sqrt{1+q^2}}(e_k\otimes e_j-qe_j\otimes e_k)\qquad (ijk)=\begin{cases}(123)\\(231)\\(312)\end{cases}\\ \bar{f}_{(kj)}&=\frac{1}{\sqrt{1+q^{-2}}}(e_k\otimes e_j+q^{-1}e_j\otimes e_k)\end{aligned} \tag{8}$$

are eigenvectors of the matrix $\hat{W}$ satisfying the eigenvalue equations

$$PR(z,q)\bar{e}_i=-(zq^{-1}-zq)\bar{e}_i$$
$$PR(z,q)\bar{f}_{(kj)}=(zq-z^{-1}q^{-1})\bar{f}_{(kj)}.$$

Therefore at $z=q^{-1}$ the matrix $\hat{W}$ becomes that of a projection:

$$PR(q^{-1},q)=(q^2-q^{-2})\sum_i\bar{e}^t_i\otimes\bar{e}_i.$$

Let us now stretch each vertex along the diagonal as shown in figure 3. We obtain a hexagonal lattice, which we shall call a decorated lattice. The side diagonal on the square lattice corresponds to the north direction on the decorated lattice. From (8) it follows that partition function (4) can be rewritten as a partition function on the hexagonal lattice. The states on the inclined edges correspond to the vectors $\bar{e}_i$. With the vertices we associate the weights $(\bar{e}_i, e_k \otimes e_j)$ and $(e_i \otimes e_j, \bar{e}_k)$ where i, j, k are the states on the neighbour edges and e_i are the vectors associated with the states on the horizontal and vertical edges. Let us represent the states on the edges by arrows, as shown in figure 4.

As a result we obtain that the partition function $(q-q^{-1})^{-N_0}\tilde{Z}_{N_0}(q^{-1}; q)$ is equal to the partition function on a hexagonal lattice with the vertices of an ice type given in figure 5.

Arrows form closed contours on the lattice. It is easy to see that on the lattice with cyclic boundary conditions the contribution of each closed contour to the partition function is positive.

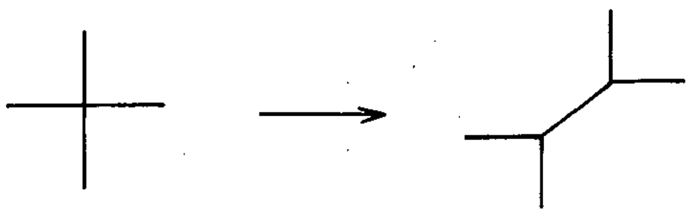

Figure 3. Decoration.

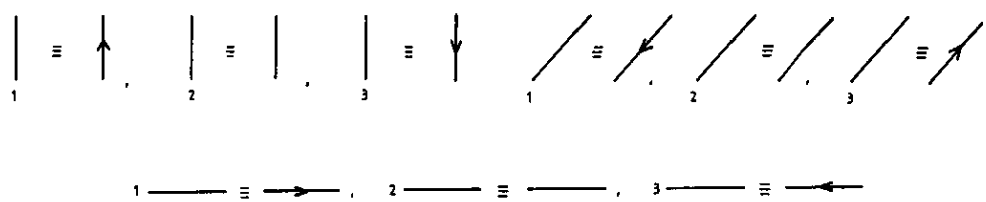

Figure 4. Arrow configurations.

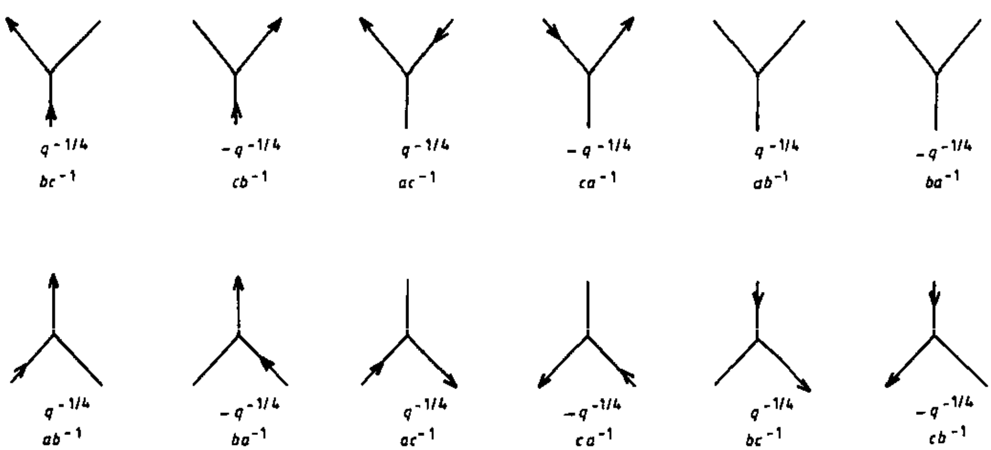

Figure 5. Boltzmann weights and gauge transformation weights (a, b, c).

Let a, b, c be arbitrary non-zero numbers. The partition function of the model on a decorated lattice does not change if we multiply the vertices by the factors shown in figure 5. Choosing $b = q^{-1/3}a$, $c = q^{-2/3}a$ we obtain the following expression for the partition function:

$$\tilde{Z}_{N_0}(q^{-1}; q) = (q - q^{-1})^{N_0} \sum_{\hat{C}} q^{(1/6)(N_R(\hat{C}) - N_L(\hat{C}))} \tag{9}$$

where the sum runs over all dense (see the introduction) oriented loops on the hexagonal lattice and $N_R(\hat{C})$ ($N_L(\hat{C})$) is the number of vertices at which two arrows form a right (left) turn.

It is easy to see that

$$N_R(\hat{C}) - N_L(\hat{C}) = 6 \sum_{i=1}^{N(\hat{C})} \varepsilon_i$$

where ε_i is an orientation of the ith loops $\varepsilon_i = +1$ (-1) for right (left) oriented loops.

Taking the summation over orientation in (6) we obtain the partition function (1) for $n = q + q^{-1}$:

$$\tilde{Z}_{N_0}(q^{-1}; q) = (q - q^{-1})^{N_0} \sum_{\hat{C}} (q + q^{-1})^{N(\hat{C})}. \tag{10}$$

To conclude this section let us note that the boundary conditions on the decorated lattice are inherited from the periodic boundary conditions on the square lattice.

3. The thermodynamic limit

In the thermodynamic limit, as $N, M \to \infty$, the asymptotics of a partition function are defined by the largest eigenvalue of the transfer matrix [12]. The eigenvalues of matrix (6) can be found easily by means of the multicomponent algebraic Bethe ansatz method [10, 13]. It is convenient to use new variables instead of z, q: $z = e^u$, $q = e^\eta$ for $n > 2$, $z = e^{i\nu}$, $q = e^{i\gamma}$ for $n < 2$. In (1), $n = 2$ corresponds to the limiting case of $z = e^{i\nu\varepsilon}$, $q = e^{i\varepsilon}$, $\varepsilon \to 0$.

For $n > 2$ we obtain with the help of the Bethe ansatz method the following expression for the eigenvalues of $t(z)$:

$$\begin{aligned}\Lambda(u, \eta) = (2\sinh(u+\eta))^N \prod_{\lambda^{(1)}} \frac{\sinh(u - i\lambda^{(1)} - \eta/2)}{\sinh(u - i\lambda^{(1)} + \eta/2)} \\ + (2\sinh u)^N \prod_{\lambda^{(1)}} \frac{\sinh(u - i\lambda^{(1)} + 3\eta/2)}{\sinh(u - i\lambda^{(1)} + \eta/2)} \prod_{\lambda^{(2)}} \frac{\sinh(u - i\lambda^{(2)})}{\sinh(u - i\lambda^{(2)} + \eta)} \\ + (2\sinh u)^N \prod_{\lambda^{(2)}} \frac{\sinh(u - i\lambda^{(2)} + 2\eta)}{\sinh(u - i\lambda^{(2)} + \eta)}.\end{aligned} \tag{11}$$

Here the numbers $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$ are solutions of the following system of Bethe equations:

$$\begin{aligned}\left(\frac{\sinh(i\lambda_j^{(1)} + \eta/2)}{\sinh(i\lambda_j^{(1)} - \eta/2)}\right)^N = \prod_{k \neq j}^{n_1} \frac{\sinh(i\lambda_j^{(1)} - i\lambda_k^{(1)} + \eta)}{\sinh(i\lambda_j^{(1)} - i\lambda_k^{(1)} - \eta)} \prod_{k=1}^{n_2} \frac{\sinh(i\lambda_j^{(1)} - i\lambda_k^{(2)} - \eta/2)}{\sinh(i\lambda_j^{(1)} - i\lambda_k^{(2)} + \eta/2)} \\ 1 = \prod_{k \neq j}^{n_2} \frac{\sinh(i\lambda_j^{(2)} - i\lambda_k^{(2)} + \eta)}{\sinh(i\lambda_j^{(2)} - i\lambda_k^{(2)} - \eta)} \prod_{k=1}^{n_1} \frac{\sinh(i\lambda_j^{(2)} - i\lambda_k^{(1)} - \eta/2)}{\sinh(i\lambda_j^{(2)} - i\lambda_k^{(1)} + \eta/2)}.\end{aligned} \tag{12}$$

The eigenvalues of $t(z, q)$ for $n < 2$ are obtained from (11) and (12) after the following change of variables:

Taking the logarithm of (12) we obtain the following system of transcendental equations:

$$N\phi(\lambda_j^{(1)},\eta)=2\pi I_j^{(1)}+\sum_{l\neq j}^{n_1}\phi(\lambda_j^{(1)}-\lambda_l^{(1)},2\eta)-\sum_{l=1}^{n_2}\phi(\lambda_j^{(1)}-\lambda_l^{(2)},\eta)$$

$$0=2\pi I_j^{(2)}+\sum_{l\neq j}^{n_2}\phi(\lambda_j^{(2)}-\lambda_l^{(2)},2\eta)-\sum_{l=1}^{n_1}\phi(\lambda_j^{(2)}-\lambda_l^{(1)},\eta) \tag{13}$$

$$\phi(u,\eta)=2\tan^{-1}\left(\tan u\coth\left(\frac{\eta}{2}\right)\right).$$

Here the numbers $I_j^{(k)}$ are integers or half-integers, $2I_j^{(1)}=N+n_1+n_2-1 \pmod 2$, $2I^{(2)}=n_1+n_2-1 \pmod 2$.

The largest eigenvalue exists only if $N\equiv 0 \pmod 3$ and corresponds to the following numbers $\{I^{(k)}\}$, n_k:

$$n_1=\frac{2N}{3} \qquad n_2=\frac{N}{3}$$

$$I_j^{(1)}=\frac{1}{2}\left(\frac{2N}{3}+1-2j\right) \qquad j=1,\ldots,\frac{2N}{3}$$

$$I_j^{(2)}=\frac{1}{2}\left(\frac{N}{3}+1-2j\right) \qquad j=1,\ldots,\frac{N}{3}.$$

As $N\to\infty$ the numbers $\lambda_j^{(k)}$ fill the real axis with densities $\rho^{(k)}(\lambda)$:

$$\lim_{N\to\infty}\frac{1}{N(\lambda_{j+1}^{(k)}-\lambda_j^{(k)})}=\rho^{(k)}(\lambda).$$

From (13) we obtain the system of integral equations for $\rho^{(k)}(\lambda)$:

$$\phi'(\lambda,\eta)=2\pi\rho^{(1)}(\lambda)+\int_{-\pi/2}^{\pi/2}(\phi'(\lambda-\mu,2\eta)\rho^{(1)}(\mu)-\phi'(\lambda-\mu,\eta)\rho^{(2)}(\mu))\,\mathrm{d}\mu$$

$$0=2\pi\rho^{(2)}(\lambda)+\int_{-\pi/2}^{\pi/2}(\phi'(\lambda-\mu,2\eta)\rho^{(2)}(\mu)-\phi'(\lambda-\mu,\eta)\rho^{(1)}(\mu))\,\mathrm{d}\mu$$

which can be solved using Fourier series, and we obtain the following expression:

$$\rho^{(k)}(\lambda)=\sum_{n=-\infty}^{+\infty}\mathrm{e}^{2\mathrm{i}\lambda n}\frac{\sinh[(3-k)n\eta]}{\sinh(3n\eta)}. \tag{14}$$

Passing in (11) to the limit as $N\to\infty$ and substituting (13) in the limit expression for the largest eigenvalue we obtain its asymptotics in the thermodynamic limit:

$$\Lambda_{\max}(q^{-1};q)=(4\sinh(3\eta)\sinh\eta\,\mathrm{e}^{-7\eta/3}F(q^{-2},q^{-6}))^N(1+\mathrm{o}(1)) \tag{15}$$

where $F(z,q)=\Pi_{n>0}(1-zq^n)/(1-q^n)$ for $|q|<1$.

From (15), (7) and (10) we find the asymptotics of the partition function (5) for $n>2$:

$$Z_{N_0}(q+q^{-1})=((q^3-q^{-3})q^{-7/3}F(q^{-2},q^{-6}))^{N_0}(1+\mathrm{o}(1)). \tag{16}$$

Similar calculations give the following asymptotics of the partition function for $n<2$:

$$Z_{N_0}(2\cos\gamma)=\exp\left(N_0\int_{-\infty}^{+\infty}\frac{\sinh((\pi-\gamma/2)x)\sinh(\gamma x/2)^2}{x\sinh(\pi x/2)\sinh(3\gamma x/2)}\mathrm{d}x\right)(1+\mathrm{o}(1))$$

and for $n=2$

$$Z_{N_0}(2)=\exp\left(2N_0\int_0^\infty \mathrm{e}^{-t/2}\frac{\sinh(t/2)^2}{t\sinh(3t/2)}\,\mathrm{d}t\right)(1+\mathrm{o}(1)).$$

So, we calculate the free energy of model (5) explicitly. From (16) we conclude that there are singularities in the expression of the free energy as $q\to 1+0$. This means that $n=2$ is the critical point for model (5).

4. Discussion of the critical behaviour

For $n>2$ there is a gap dividing the largest eigenvalue of the transfer matrix from low-lying eigenvalues in the thermodynamic limit [10]. This means that for $n>2$ we have a finite correlation length which is inversely proportional to the gap. The correlation length ξ grows to infinity in the vicinity of the critical point $n_c=2$

$$\xi\sim\exp\left(\frac{\pi^2}{3|n-2|^{1/2}}\right).$$

Therefore $n=2$ is a critical point of the Kosterlitz–Thouless type.

For $n\leqslant 2$ there is no gap in the spectrum of transfer matrix in the thermodynamic limit and, moreover, the dispersions have a sound-type spectrum near the Fermi surface. This means that the leading asymptotics of correlation functions must be scale invariant, and can be described by a two-dimensional effective conformal field theory. The central charge c and the spectrum of anomalous dimensions Δ_j of this conformal field theory can be found by analysing finite-size corrections [14] of the spectrum of the transfer matrix (3). The calculation of these finite-size corrections is similar to that given in [15–17]. The answer is that the central charge c of the effective conformal field theory does not depend on γ, and $c=2$. The spectrum of anomalous dimensions is of the form

$$\Delta^{(\pm)}=\tfrac{1}{4}\sum_{ji=1}^{2}(l_i a_{ij} l_j+d_i(a^{-1})_{ij}d_j)\pm\sum_i \tfrac{1}{2}l_i d_i+I^{(\pm)}$$

$$a_{ij}=\frac{\pi}{\pi-\gamma}c_{ij}$$

where l_i, d_i, $i=1,2$, and $I^{(\pm)}\geqslant 0$ are integers, and C is the Cartan matrix for sl_3:

$$C=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\qquad C^{-1}=\frac{1}{3}\begin{pmatrix}2&1\\1&2\end{pmatrix}.$$

Here we consider the cyclic boundary conditions on the square lattice. This is an unnatural boundary condition for the hexagonal lattice, which is a decorated square lattice. But for $n>2$ the correlation length is finite and the theory in the thermodynamic limit does not depend on boundary conditions (if, of course, they are not orthogonal to the ground state of the model). For $n\leqslant 2$ the boundary conditions can change the thermodynamic limit of the partition function and of correlators. It will be interesting to understand this case in greater detail.

Let us compare the exact solution of model (5) with known results about the O(n)-model. As follows from the result given here and from Baxter's solution, $n=2$ is the point of phase transition. For $n>n_c$ the model has a finite correlation length and for $n\leqslant n_c$ the model is critical. This fact is in good agreement with the results

[1-3]. But from renorm-group calculations and from the results obtained for the model on random lattices it follows that for $T \neq 0$ the critical O(n)-model is described by minimal conformal theories [18-20]. If these results could be continued on the line $T = 0$ they would contradict the above results. The finite-size analysis of the O(n)-model along Baxter's line [21-23] gives $C = 1 - 6(g-1)^2/g$ where $n = -2\cos\pi g$. However, it cannot be continued at the line $T = 0$.

5. Conclusion

We have considered the loop model connected with the sl_3 R-matrix with anisotropy. The method of decorating the lattice in accordance with the structure of the R-matrix seems universal. Using this method one can associate with any R-matrix some models, not only on a square lattice but on some decorated lattices. It will be interesting to find any reasonable decorations of models connected with R-matrices corresponding to Kac-Moody algebras [24].

Acknowledgments

I would like to thank V Jones, V Kazakov and F Smirnov for discussions and the University of California at Berkeley and MSRI for hospitality.

Appendix

Baxter rewrites (2) as the partition function on a hexagonal lattice with the ice-type vertices shown in figure 6. To compare this model with the model given in [5] let us transform the hexagonal lattice into the square one using summation over 'intermediate

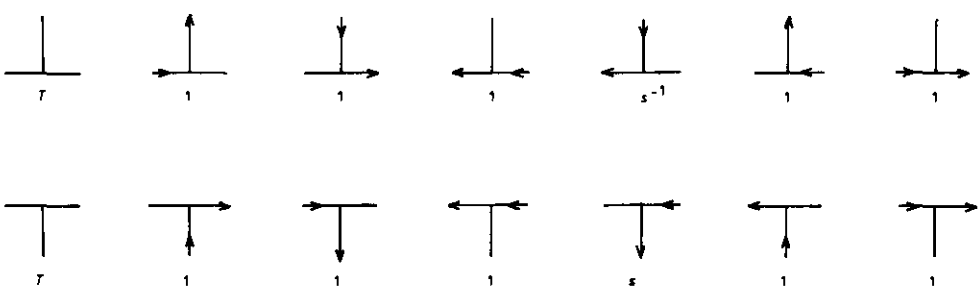

Figure 6. Boltzmann weights in the Baxter model.

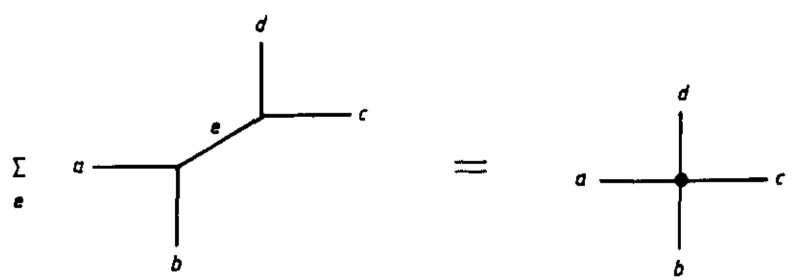

Figure 7. Transformation of the hexagonal lattice to the square one.

states' (figure 7). Next, let us enumerate the states on the edges by the basic vectors in $\mathbb{C}^3$ (see figure 4). We obtain a model on the square lattice whose matrix of the vertex weights (figure 1) is of the form

$$W(n)=\left|\begin{array}{ccc|ccc|ccc} 0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 0&0&1&0&T&0&1&0&0\\ \hline 0&1&0&1&0&0&0&0&0\\ 0&0&T&0&T^2&0&T&0&0\\ 0&0&0&0&0&1&0&s&0\\ \hline 0&0&1&0&T&0&1&0&0\\ 0&0&0&0&0&s^{-1}&0&1&0\\ 0&0&0&0&0&0&0&0&0 \end{array}\right|. \tag{A1}$$

Here $s+s^{-1}=4T^2-2-T^4=n$. The matrix of the vertex weights in the model related to the Kac–Moody algebra $A_2^{(2)}$, also known as the Izergin–Korepin (IZ) model, have the following non-zero elements:

$$\begin{aligned}
&R_{11}^{11}(\lambda)=R_{33}^{33}(\lambda)=\lambda^5p^5-\lambda^{-1}p^{-5}+p^{-1}-p\\
&R_{12}^{12}(\lambda)=R_{21}^{21}(\lambda)=R_{23}^{23}(\lambda)=R_{32}^{32}(\lambda)=\lambda p^3-\lambda^{-1}p^{-3}+p^{-3}-p^3\\
&R_{13}^{13}(\lambda)=R_{31}^{32}(\lambda)=\lambda p-\lambda^{-1}p^{-1}+p^{-1}-p\\
&R_{22}^{22}(\lambda)=\lambda p^3-\lambda^{-1}p^{-3}+p^{-3}-p^3+p^{-1}-p+p^5-p^{-5}\\
&R_{21}^{12}(\lambda)=R_{32}^{23}(\lambda)=\lambda(p^5-p)+p^{-1}-p^{-5}\\
&R_{22}^{13}(\lambda)=R_{31}^{22}(\lambda)=\lambda(p^4-1)+1-p^4\\
&R_{13}^{13}(\lambda)=\lambda(p^5-p-p^3+p^{-1})+p^3-p^{-5}\\
&R_{12}^{21}(\lambda)=R_{23}^{32}(\lambda)=\lambda^{-1}(p^{-1}-p^{-5})+p^5-p\\
&R_{13}^{22}(\lambda)=R_{22}^{31}(\lambda)=\lambda^{-1}(1-p^{-4})+p^{-4}-1\\
&R_{13}^{31}(\lambda)=\lambda^{-1}(p^{-1}-p^{-5}-p+p^{-3})+p^5-p^{-3}.
\end{aligned} \tag{A2}$$

It is straightforward to verify the following relation between matrices (A1) and (A2):

$$W(-\mathrm{i}(p-p^{-1}))=(V^{-1}S\otimes V)R(p^{-4};p)(S^{-1}V\otimes V^{-1})$$

where $s=-p^4$, $T=-\mathrm{i}(p-p^{-1})$ is the uniformization of curve $s^{-1}+s=2T-2-T^4$ and

$$S=\begin{pmatrix}\mathrm{i}&0&0\\0&1&0\\0&0&-\mathrm{i}\end{pmatrix}\qquad V=\begin{pmatrix}1&0&0\\0&y&0\\0&0&1\end{pmatrix}\qquad y^2=\mathrm{i}p^2.$$

So, we have

$$Z_{\text{loop}}(-\mathrm{i}(p-p^{-1}),-(p^4+p^{-4}))=Z_{\text{IZ}}(p^{-4};p)$$

where $Z_{\text{IZ}}(\lambda;p)$ is defined by (A2), (6) and (7).

We proved the equivalence of the models (A2) and (2), (3) by transforming the hexagonal lattice into the square lattice. One can prove this equivalence in the opposite direction. For this one has to use the decoration of the square lattice (2) with the help of matrices (A2) at $\lambda=p^{-4}$ where its rank is 3.

The eigenvalues of row-to-row transfer matrices of the IK model were found in [7] and are of the following form:

$$\Lambda(\lambda)=(R_{11}^{11}(\lambda))^N\prod_{j=1}^{n}\frac{\lambda-\mu_j p^2}{\lambda p^2-\mu_j}+(R_{12}^{12}(\lambda))^N\prod_{j=1}^{n}\frac{\lambda p^5-\mu_j}{\lambda p^2-\mu_j}\frac{\lambda+\mu_j}{\lambda p^4+\mu_j}$$
$$+(R_{13}^{13}(\lambda))^N\prod_{j=1}^{n}p^{-2}\frac{\lambda p^8+\mu_j}{\lambda p^4+\mu_j} \tag{A3}$$

where the μ_j satisfy the equation

$$\left(\frac{\mu_k p^2-1}{\mu_k-p^2}\right)^N=\prod_{j\neq k}^{n}\frac{\mu_k p^4-\mu_j}{\mu_k-\mu_j p^4}\frac{\mu_k+\mu_j p^2}{\mu_k p^2+\mu_j}. \tag{A4}$$

Substituting $\lambda=p^{-4}$ we obtain Baxter's result. The difference between approaches [5] and [7] is that Baxter found the coordinate eigenvectors of the transfer matrix and in [7] the eigenvalues were found from the system of functional equations. The algebraic form of eigenvectors in the IK model was given by Tarasov [25].

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